NYO-1480-54
MF-50

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Mathematical Sciences

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AEC Research and Development Report

TID-4500
PHYSICS
April 1, 1966

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A condensed version of this report will appear in the Proceedings of the Seventh International Conference on Phenomena in Ionized Gases held in Belgrade, Yugoslavia on 22-27 August 1965.

Contract No. AT(30-1)-1480

Abstract

We describe the speed and structure of steady nonlinear waves moving with uniform velocity in an oblique direction across a magnetic field in a collision-free plasma with temperature. This is a self-consistent formulation in which the plasma motion is governed by the Vlasov equations for ions and electrons coupled to the Poisson-Maxwell equations. This difficult problem can be solved in the limiting case of small-amplitude nonlinear waves if we assume that the same scaling laws hold for the warm plasma as those for a cold plasma.

Acknowledgement

This work was carried out while one of the authors (H.K.) was on leave of absence from the Institut f. Plasmaphysik, Jülich, Assoziation KFA-Euratom. He would like to thank members of the Courant Institute of Mathematical Sciences for their hospitality.

We thank Eva V. Swenson for handling the numerical calculations.

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1. Introduction

The effect of temperature on the structure of nonlinear steady plane hydromagnetic waves which can propagate transverse to the magnetic field in a collision-free plasma has been studied in [1]. This is a self-consistent formulation in which the equations of motion (Vlasov equations) for ions and electrons are coupled with the Poisson and Maxwell equations. In the limiting case of a wave of small amplitude, a particular nonlinear progressing-wave solution in the form of a single pulse is obtained by using the same scaling laws which hold for the solitary wave in a cold plasma derived in [2].

In this paper, we summarize our results leading to a description of the speed and structure of steady nonlinear waves moving with uniform velocity in an oblique direction across a magnetic field in a collision-free plasma with finite temperature. Again we assume that the same scaling laws hold for the warm plasma as those for a cold plasma; for the oblique case the cold-plasma scaling laws are given

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1. C. S. Gardner and G. K. Morikawa, "Effect of Temperature on the Width of a Small-amplitude Solitary Wave in a Collision-free Plasma," Comm. Pure and Appl. Math. 18, 35-49 (1965).

2. C. S. Gardner and G. K. Morikawa, "Similarity in the Asymptotic Behavior of Collision-free Hydromagnetic Waves and Water Waves," New York University, Courant Institute of Mathematical Sciences, Report TID-6184 (May 1960).

in [3]. Compared to the transverse case, the distribution functions for ions and electrons have a considerably greater variation in structure in the oblique case. As in the cold-plasma oblique case, there exists a critical transition angle at which the steady wave shape changes from a positive (compression) pulse into a negative (expansion) pulse; for the warm plasma this transition angle depends on the temperatures of both the ions and electrons as well as on the mass ratio of electron to ion. In contrast to the cold-plasma oblique case, singularities in the higher-order perturbed states of the distribution functions arise in velocity space. These singularities are related to those particles which move with the wave in the direction of the unperturbed magnetic field. Consequently there is a nonuniqueness in the specification of both the speed and structure of the wave. Apparently we must study more extensive problems which include a greater variety of motions in order to completely resolve this difficulty, e.g., time-dependent problems such as in [2] and [3].

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3. K. W. Morton, "Finite Amplitude Compression Waves in a Collision-free Plasma", Phys. Fluids 7, 1800-1815 (1964).

2. Scaling Transformations and Equations of Motion

We consider a plane plasma wave moving with uniform speed U_0 in the negative $\overline{x}$ direction. The undisturbed magnetic field ahead of the wave has constant strength B_0 directed at an angle α with respect to the wave front; α is in the $\overline{x}$-$\overline{z}$ plane (Fig. 1). We formulate the problem in the frame of reference fixed to the wave; in the wave frame there is a constant electric field in the $\overline{y}$-direction of magnitude $E_0 = U_0 B_0$. Then the magnetic and electric field components may be written

$$(B_x, B_y, B_z) = B_0(\sin\alpha,\ b,\ \cos\alpha + B) \tag{1}$$

and

$$(E_x, E_y, E_z) = U_0 B_0(E,\ \cos\alpha,\ 0) \tag{2}$$

where b, B, and E are the perturbed field components.

We look for nontrivial solutions which go to a constant state at $x = -\infty$, where the distribution functions $\overline{f}_+$ for ions and $\overline{f}_-$ for electrons become maxwellian in the undisturbed plasma and the plasma is charge-neutral, $n_- = n_+ = n_0$. The overall procedure for solving this problem is similar to that described in [1] for the transverse-magnetic-field case. In the oblique case we also make a coordinate transformation in velocity space to facilitate the integration of the Vlasov equations, i.e., cylindrical polar coordinates (θ, V, w) in which the w-axis is aligned with the direction of the magnetic field B_0 (see Fig. 2):

$$\overline{u} = U_0(1 + u \cos\alpha + w \sin\alpha) \tag{3.1}$$

$$\overline{v} = U_0 v \tag{3.2}$$

$$\overline{w} = U_0(-u \sin\alpha + w \cos\alpha) \tag{3.3}$$

where

$$u = V \cos\theta , \qquad v = V \sin\theta .$$

Guided by the asymptotic method of solution in [1], [2] and [3] we define a small parameter ε (related to the strength of the wave) in terms of a Mach number:

$$M^2 = \frac{U_0^2}{A^2 + a^2} = 1 \pm \varepsilon^2 \tag{4.1}$$

where A is the Alfvén speed,

$$A^2 = \frac{B_0^2}{\mu n_0(m_+ + m_-)} \tag{4.2}$$

and a is akin to the sound speed and must be determined as part of the solution. In the transverse magnetic field limit, $\alpha = 0$, we must get

$$a^2 = \frac{2(p_+ + p_-)}{n_0(m_+ + m_-)} \tag{5}$$

appropriate for a plasma with specific heat ratio, $c_p/c_v = 2$. In the definition of ε in (4.1) we consider the possibility of having both subcritical $(1-\varepsilon^2)^{\frac{1}{2}}$ and supercritical $(1+\varepsilon^2)^{\frac{1}{2}}$ wave speeds, M.

We now assume that the same scaling laws and expansion of the field quantities (with respect to ε) hold for the warm plasma as for the cold plasma [3]:

$$\bar{x} = \frac{1}{\varepsilon}\frac{m_+ U_0}{eB_0} x \tag{6}$$

where (m_+U_0/eB_0) is the ion gyro radius and the perturbed field expansions of the field components B_y, B_z, and E_x, respectively, are

$$b = \varepsilon^3 b^{(1)} + \varepsilon^5 b^{(2)} + O(\varepsilon^7) \tag{7.1}$$

$$B = \varepsilon^2 B^{(1)} + \varepsilon^4 B^{(B)} + O(\varepsilon^6) \tag{7.2}$$

$$E = \varepsilon^3 E^{(1)} + \varepsilon^5 E^{(2)} + O(\varepsilon^7) \ . \tag{7.3}$$

The superscript quantities are independent of ε. We introduce dimensionless functions f_+ and f_- by

$$\bar{f}_+ = \frac{n_0}{U_0^3} f_+ \quad \text{and} \quad \bar{f}_- = \frac{n_0}{U_0^3} f_- \tag{8}$$

We need only work with the distribution function f_+ for ions since the Vlasov equations show that f_- can be obtained from f_+ by replacing terms with $\partial(\ )/\partial x$ in f_+ by $(-m)\partial(\ )/\partial x$ and terms with $\int(\ )dx$ in f_+ by $(-m^{-1})\int(\ )dx$ where $m = m_-/m_+$ is the mass ratio of electron to ion. We drop the subscript on $f_+ \equiv f$ and the Vlasov equation of motion for ions can be written

$$\frac{\partial f}{\partial \theta} = \varepsilon(1 + w \sin \alpha + V \cos \theta \cos \alpha) \frac{\partial f}{\partial x} + E_{\parallel} \frac{\partial f}{\partial w} \tag{9.1}$$

$$+ \left\{ E_{\perp} \frac{\partial}{\partial V} + b(V \frac{\partial}{\partial w} - w \frac{\partial}{\partial V}) + \frac{(E_{\perp} - wb)}{V} \right\} (f \cos \theta)$$

$$- B\left\{\frac{\partial}{\partial V} + \sin \alpha (w \frac{\partial}{\partial V} - V \frac{\partial}{\partial w}) + \frac{(1 + w \sin \alpha)}{V} \right\} (f \sin \theta)$$

$$+ \frac{\partial}{\partial \theta}\left\{ \frac{(wb - E_{\perp})}{V} (f \sin \theta) - Bf \cos \alpha \right.$$

$$\left. - \frac{B(1 + w \sin \alpha)}{V} (f \cos \theta) \right\}$$

where the notation $E_{\parallel}$ and $E_{\perp}$ have been introduced to simplify the equation,

$$E_{\parallel} \equiv E \sin \alpha + b \cos \alpha \quad \text{and} \quad E_{\perp} = E \cos \alpha - b \sin \alpha \ . \tag{9.2}$$

Recalling the expansion (7) we see that the right hand side of (9.1) is $O(\varepsilon)$.

With the velocity-space transformation (3) and scaling transformations (3) to (6) the Poisson equation becomes

$$\varepsilon(1 + m) \frac{A^2}{c^2} \frac{dE}{dx} = \int_0^{\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} (f_+ - f_-) V \, dV \, d\theta \, dw \tag{10.1}$$

$$= N(x)$$

where $m = (m_-/m_+)$ and c is the light speed. By (1) to (6) the Maxwell equations become

$$\varepsilon(1+m)[1-(1+\varepsilon^2)\frac{a^2}{U_0^2}]\frac{db}{dx} = (1+\varepsilon^2)\int_0^\infty\int_0^{2\pi}\int_{-\infty}^\infty(-V\cos\theta\sin\alpha$$

$$+ w\cos\alpha)(f_+ - f_-)V\,dV\,d\theta\,dw \tag{10.2}$$

$$= (1+\varepsilon^2)j(x)$$

and

$$\varepsilon(1+m)(1-(1+\varepsilon^2)\frac{a^2}{U_0^2}]\frac{dB}{dx}$$

$$= (1+\varepsilon^2)\int_0^\infty\int_0^{2\pi}\int_{-\infty}^\infty(V\sin\theta)(f_+ - f_-)V\,dV\,d\theta\,dw \tag{10.3}$$

$$= (1+\varepsilon^2)J(x) \ .$$

For later sections we use the abbreviated notations:

N is the excess number density of singly charged ions to electrons;

j is the z-component of the current density; and

J is the y-component of the current density.

3. Series Solution of the Vlasov (Collision-Free Boltzmann) Equations

We assume that the distribution functions $f \equiv f_+$ and f_- can be represented as power series expansions in ε, as in [1]:

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4 + \varepsilon^5 f_5 + O(\varepsilon^6) \tag{11}$$

and similarly for f_-. f_0 is prescribed to be Maxwellian. If now we had followed the algorithm for determining f_k in [1], we would insert (11) into (9) and equate coefficients of like powers in ε; then we would 1) integrate the equation for $\partial f_k/\partial\theta$ beginning with $k = 1$, and 2) use a compatibility condition for the solvability of the next-order equation for f_{k+1}, i.e., integrate the equation for $\partial f_{k+1}/\partial\theta$ from $\theta = 0$ to $\theta = 2\pi$, in order to evaluate the arbitrary θ-integration variable for f_k in 1). This procedure is correct in principle but unnecessarily tedious. A much simpler, but equivalent, algorithm for determining f_k is available by recognizing the unique functional form of f_k with respect to θ, namely,

$$f_k = A_{k0} + \sum_{\ell=1}^{k-1} (A_{k\ell} \cos \ell\theta + B_{k\ell} \sin \ell\theta) \tag{12}$$

and obtaining (from the Vlasov equation) recurrence relations relating the coefficients $A_{k\ell}$, $B_{k\ell}$, and A_{k0}, which are functions of V, w, the perturbed fields and their derivatives, to the lower order coefficients.

We prescribe the unperturbed distribution function f_0 to be maxwellian

$$f_0 = A_{00} = \Psi(V^2 + w^2) \tag{13.1}$$

where

$$\Psi = \Psi_+ = (\gamma/\pi)^{3/2} \exp[-\gamma(V^2 + w^2)] \;, \quad \gamma_+ = \frac{m_+ U_0^2}{2kT_+} \equiv \gamma \tag{13.2}$$

For electrons the unperturbed distribution function is

$$\Psi_- = (\gamma_-/\pi)^{3/2} \exp[-\gamma_-(V^2 + w^2)] \;, \quad \gamma_- = \frac{m_- U_0^2}{2kT_-} \;. \tag{13.3}$$

We note that a more general anisotropic distribution f_0, in place of (13.1), is allowed, i.e., a monotonically decreasing function of $(V^2 + \lambda w^2)$, $\lambda > 0$. The perturbed f_k's ($k = 1,2,3,4$) which we need in the later sections are the following:

$$f_1 = 0 \tag{14}$$

$$f_2 = A_{20} + A_{21} \cos\theta \tag{15.1}$$

$$A_{20} = -\frac{\Psi'}{W}\left(B^{(1)} V^2 \cos\alpha + 2\,\Phi_{\shortparallel}^{(1)} w\right), \quad \Phi_{\shortparallel}^{(j)} = \int_{-\infty}^{x} E_{\shortparallel}^{(j)} dx \tag{15.2}$$

$$A_{21} = 2B^{(1)}\,\Psi' V \tag{15.3}$$

where $W = (1 + w \sin\alpha)$ and primes indicate derivatives and

$$f_3 = B_{31}\sin\theta + B_{32}\sin 2\theta; \quad A_{3\ell} = 0, \quad \ell = 0,1,2,\ldots \tag{16.1}$$

$$B_{31} = B'^{(1)}\Psi' V\left(2W - \frac{V^2\cos^2\alpha}{W}\right) + \frac{2(E_\perp^{(1)} - b^{(1)}w)\Psi' V}{W} \tag{16.2}$$

$$B_{32} = \frac{1}{2} B'^{(1)}\Psi' V^2\cos\alpha \tag{16.3}$$

$$f_4 = A_{40} + A_{41}\cos\theta + A_{42}\cos 2\theta + A_{43}\cos 3\theta \tag{17.1}$$

$$B_{4\ell} = 0\ , \quad \ell = 0,1,2,3,\ldots$$

$$\begin{aligned} A_{40} &= B''^{(1)}\Psi' V^2\cos\alpha\left(W - \frac{3}{8}\frac{V^2\cos^2\alpha}{W}\right) \\ &+ \frac{(E_\perp'^{(1)} - b'^{(1)}w)\Psi' V^2\cos\alpha}{W} + (B^{(1)})^2\Big\{\Psi'' V^2\left(1 + \frac{V^2\cos^2\alpha}{2W^2}\right) \\ &+ \Psi'\left[W + \frac{V^2(1 - \frac{3}{2}\sin\alpha)}{W} + \frac{V^4\sin^2\alpha\cos^2\alpha}{4W^3}\right]\Big\} \\ &+ (B^{(1)}\Phi_{\shortparallel}^{(1)})\frac{V^2\cos\alpha}{W}\frac{\partial}{\partial w}\left(\frac{\Psi'}{W}\right) \\ &+ 2\left(\int_{-\infty}^{x} E_{\shortparallel}^{(1)}\Phi_{\shortparallel}^{(1)}dx\right)\frac{1}{W}\frac{\partial}{\partial w}\left(\frac{\Psi' w}{W}\right) \\ &- \frac{\Psi'}{W}\left(B^{(2)}V^2\cos\alpha + 2\Phi_{\shortparallel}^{(2)}w\right)\ . \end{aligned} \tag{17.2}$$

$$A_{41} = -B''^{(1)}\Psi' V(2W^2 - \tfrac{3}{4}V^2\cos^2\alpha) - 2(E_\perp'^{(1)} - b'^{(1)}w)\Psi' V$$

$$- (B^{(1)})^2 V\cos\alpha(4\Psi' + \frac{2\Psi'' V^2}{W} + \frac{\Psi' V^2 \sin^2\alpha}{W^2}) \tag{17.3}$$

$$- 2B^{(1)}\Phi_{\shortparallel}^{(1)} V \frac{\partial}{\partial w}(\frac{\Psi'}{W}) + 2B^{(2)}\Psi' V$$

$$A_{42} = -\frac{1}{4}B''^{(1)}\Psi' V^2\cos\alpha(3W - \frac{V^2\cos^2\alpha}{W}) + (B^{(1)})^2\Psi'' V^2$$

$$- \frac{1}{2}(E_\perp'^{(1)} - b'^{(1)}w)\frac{\Psi' V^2\cos\alpha}{W} \tag{17.4}$$

$$A_{43} = -\frac{1}{12}B''^{(1)}\Psi' V^3\cos^2\alpha \tag{17.5}$$

By inspection, (13) to (17) show that f_k can be distinguished between even and odd k. For even k

$$f_{2k} = A_{2k,0} + \sum_{\ell=1}^{2k-1} A_{2k,\ell}\cos\ell\theta\;, \quad f_0 = A_{00} \tag{18.1}$$

and for odd k

$$f_{2k+1} = \sum_{\ell=1}^{2k} B_{2k+1,\ell}\sin\ell\theta\;, \qquad f_1 = 0 \tag{18.2}$$

These particular forms (18) for f_k are useful in the next section in determining the expansions of the moments of f_+ and f_- in powers of ε. We point out that the singular integrals which arise when we calculate the moments of f_+ and f_- come from the zeros of powers of $W = (1 + w\sin\alpha)$ occurring in the denominator.

4. Computation of the Moments: Number Density N and y- and z-Components of Current Density J and j

Because of the forms (18) of the perturbed distribution function f_k for even and odd k, the moments N, j, and J in (10) have the following expansions with respect to ε: Only f_{2k} contribute to the number density N,

$$N = \varepsilon^2 N^{(1)} + \varepsilon^4 N^{(2)} + O(\varepsilon^6) \tag{19.1}$$

where $N^{(0)} = 0$ since the undisturbed plasma ($x = -\infty$) is charge-neutral; and also only f_{2k} contribute to the z-component of current density j,

$$j = \varepsilon^2 j^{(1)} + \varepsilon^4 j^{(2)} + O(\varepsilon^6). \tag{19.2}$$

Only f_{2k+1} contribute to the y-component of current density J,

$$J = \varepsilon^3 J^{(1)} + \varepsilon^5 J^{(2)} + O(\varepsilon^7)\ . \tag{19.3}$$

By the perturbed field expansions (7) and the perturbed density expansions (19), the Poisson-Maxwell equations (10) become

$$\varepsilon^4 (1 + m) \frac{A^2}{c^2} \frac{dE^{(1)}}{dx} = \varepsilon^2 (N^{(1)} + \varepsilon^2 N^{(2)}) + O(\varepsilon^6) \tag{20.1}$$

$$\varepsilon^4(1+m)[1-(1+\varepsilon^2)\frac{a^2}{U_0^2}]\frac{db^{(1)}}{dx} \tag{20.2}$$

$$= \varepsilon^2(1+\varepsilon^2)(j^{(1)} + \varepsilon^2 j^{(2)}) + O(\varepsilon^6)$$

$$\varepsilon^3(1+m)[1-(1+\varepsilon^2)\frac{a^2}{U_0^2}]\frac{d}{dx}(B^{(1)} + \varepsilon^2 B^{(2)}) \tag{20.3}$$

$$= \varepsilon^3(1+\varepsilon^2)(J^{(1)} + \varepsilon^2 J^{(2)}) + O(\varepsilon^7) .$$

Now the perturbed quantities with superscripts are independent of ε and we can equate coefficients of equal powers in ε in (20). To lowest order in ε we get certain consistency relations including the speed of a weak steady wave in the limit $\varepsilon \rightarrow 0$. To the next order in ε we get differential equations describing the wave structure.

5. Consistency Relations and Speed of a Weak Wave

To the lowest order in ε the Poisson-Maxwell equations (20) yield to $O(\varepsilon^2)$,

$$N^{(1)} = 0 \tag{21.1}$$

$$j^{(1)} = 0 \tag{22.1}$$

and to $O(\varepsilon^3)$,

$$(1+m)(1 - \frac{a^2}{U_0^2})\, B'^{(1)} = J^{(1)} \ . \tag{23.1}$$

By (10.1) and (19.1), the first order number density $N^{(1)}$ is

$$\begin{aligned} N^{(1)} &= \iiint (f_2^+ - f_2^-) V\, dV\, d\theta\, dw \\ &= \iiint (A_{20}^+ - A_{20}^-)\, V\, dV\, d\theta\, dw \ . \end{aligned} \tag{21.2}$$

Carrying out the integration with A_{20} given in (15), (21) states

$$\begin{aligned} B^{(1)} \sin\alpha \cos\alpha(\phi_1^+ - \phi_1^-) + 2\Phi_{\shortparallel}^{(1)}[\gamma_+(1 - \phi_1^+) \\ + m^{-1}\gamma_-(1 - \phi_1^-)] = 0 \end{aligned} \tag{21.3}$$

where ϕ_1 is the principal part of a singular integral,

$$\begin{aligned} \phi_1 &= \pi^{-\frac{1}{2}} \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-\omega^2}}{(1+\kappa\omega)}\, d\omega \ , \qquad \kappa^2 = \gamma^{-1}\sin^2\alpha \\ &= \frac{2}{\kappa} \exp(-1/\kappa^2) \int_0^{1/\kappa} \exp(\sigma^2) d\sigma \end{aligned} \tag{24.1}$$

For p^{th} order singularities we define ϕ_p as

$$\phi_p = \pi^{-\frac{1}{2}} \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{-\omega^2}}{(1+\kappa\omega)^p} \, d\omega \tag{24.2}$$

By (10.2) and (19.2) the first order current density $j^{(1)}$ is

$$\begin{aligned} j^{(1)} &= \iiint (-V \cos\theta \sin\alpha + w \cos\alpha)(f_2^+ - f_2^-) V \, dV \, d\theta \, dw \\ &= \iiint (-\tfrac{1}{2} A_{21}^+ V \sin\alpha + A_{20}^+ w \cos\alpha) \\ &\quad - (-\tfrac{1}{2} A_{21}^- V \sin\alpha + A_{20}^- w \cos\alpha)] V \, dV \, d\theta \, dw \ . \end{aligned} \tag{22.2}$$

Carrying out the integration, (22) states

$$\begin{aligned} & B'^{(1)} \sin\alpha \cos\alpha (\phi_1^+ - \phi_1^-) \\ & + 2E_{\parallel}^{(1)} [\gamma_+ (1 - \phi_1^+) + m^{-1} \gamma_- (1 - \phi_1^-)] = 0 \end{aligned} \tag{22.3}$$

which is identical to the x-derivative of (21.3). Thus $j^{(1)} = 0$ is consistent with $N^{(1)} = 0$ but gives no new information.

By (10.3) and (19.3) the first order current density $J^{(1)}$ is

$$\begin{aligned} J^{(1)} &= \iiint (V \sin\theta)(f_3^+ - f_3^-) V \, dV \, d\theta \, dw \\ &= \tfrac{1}{2} \iiint (B_{31}^+ V^2 - B_{31}^- V^2) dV \, d\theta \, dw \ . \end{aligned} \tag{23.2}$$

Carrying out the integration with B_{31} given in (16), $J^{(1)}$ is

$$J^{(1)} = -B'^{(1)}[(1+m) - (\frac{\phi_1^+}{\gamma_+} + m\,\frac{\phi_1^-}{\gamma_-})\cos^2\alpha] \tag{23.3}$$

$$-E_{\shortparallel}^{(1)}(\phi_1^+ - \phi_1^-)\,\frac{\cos\alpha}{\sin\alpha}\,.$$

Since (21.3) gives a linear relation between $E_{\shortparallel}^{(1)}$ and $B'^{(1)}$, (23.1) and (23.3) yield a relationship independent of perturbed field quantities:

$$(1+m)\,\frac{a^2}{U_0^2} = \left\{(\frac{\phi_1^+}{\gamma_+} + m\,\frac{\phi_1^-}{\gamma_-}) + \frac{1}{2}\left[\frac{(\phi_1^+ - \phi_1^-)^2}{\gamma_+(1-\phi_1^+) + m^{-1}\gamma_-(1-\phi_1^-)}\right]\right\}\cos^2\alpha \tag{25}$$

This consistency condition determines the "sound" speed a in (4) and relates it to the other physical parameters α, m, U_0, T_-, and T_+. In the limit $\varepsilon \longrightarrow 0$ in (4), the weak wave speed is given by

$$\frac{U_0^2}{A^2 + a^2} = 1 \tag{26}$$

combined with (25). Eqs. (25) and (26) recover the results in [1], [2], and [3] for the limiting cases of 1) transverse magnetic field, $\alpha = 0$, and 2) the cold plasma, $T_- = T_+ = 0$.

6. Differential Equations for the First Order Field ($E^{(1)}$, $b^{(1)}$, $B^{(1)}$)

From the Poisson-Maxwell equations (20) the system of equations which we get to the next significant order in ε are three differential equations for six unknown field quantities ($E^{(1)}$, $E^{(2)}$, $b^{(1)}$, $b^{(2)}$, $B^{(1)}$, $B^{(2)}$). But because of the asymptotic nature of our approximation procedure the second order fields ($E^{(2)}$, $b^{(2)}$, $B^{(2)}$) can be eliminated by the consistency relations (21.3) and (25). Hence we obtain three differential equations for only the first order fields ($E^{(1)}$, $b^{(1)}$, $B^{(1)}$). This system of three equations can be reduced further to a single differential equation for one field component, say $B^{(1)}$. Since the complete equations are quite lengthy we will only show the essential elements in the following description and not the explicit details.

The Poisson equation (20.1) gives to $O(\varepsilon^4)$:

$$(1+m)\,\frac{A^2}{c^2}\,E'^{(1)} = N^{(2)} \tag{27.1}$$

where

$$N^{(2)} = \iiint (f_4^+ - f_4^-)V\,dV\,d\theta\,dw \tag{27.2}$$

$$= \iiint (A_{40}^+ - A_{40}^-)V\,dV\,d\theta\,dw$$

$$= q_1 B''^{(1)} + \frac{q_2}{2}\,(B^{(1)})^2 + q_3 E'^{(1)} + q_4 b'^{(1)}$$

$$+ \frac{1}{\cos\alpha}\Big\{B^{(2)}(\phi_1^+ - \phi_1^-)\cos^2\alpha + 2\Phi_{\shortparallel}^{(2)}\,[\gamma_+(1-\phi_1^+)$$

$$+ m^{-1}\gamma_-(1-\phi_1^-)]\,\frac{\cos\alpha}{\sin\alpha}\Big\}\quad ,$$

where q_1, q_2, q_3, and q_4 have known dependence on the prescribed physical parameters (α, m, U_0, T_-, T_+); in particular, $q_3 = 0$.

The Maxwell equation (20.2) for the z-component of the current density j gives to $O(\varepsilon^4)$:

$$(1+m)(1-\frac{a^2}{U_0^2})\, b'^{(1)} = j^{(2)} \tag{28.1}$$

where

$$\begin{aligned} j^{(2)} &= \iiint (-V\cos\theta \sin\alpha + w\sin\alpha)(f_4^+ - f_4^-)V\, dV\, d\theta\, dw \qquad (28.2) \\ &= \iiint [(-\frac{A_{41}^+}{2} V\sin\alpha + A_{40}^+ w\cos\alpha) \\ &\quad - (-\frac{A_{41}^-}{2} V\sin\alpha + A_{40}^- w\sin\alpha)]V\, dV\, d\theta\, dw \\ &= r_1 B''^{(1)} + \frac{r_2}{2}(B^{(1)})^2 + r_3 E'^{(1)} + r_4 b'^{(1)} \\ &\quad - \frac{1}{\sin\alpha}\Big\{B^{(2)}(\phi_1^+ - \phi_1^-)\cos^2\alpha + 2\phi_{''}^{(2)}[\gamma_+(1-\phi_1^+) \\ &\quad + m^{-1}\gamma_-(1-\phi_1^-)]\,\frac{\cos\alpha}{\sin\alpha}\Big\} \end{aligned}$$

where r_1, r_2, r_3, and r_4 depend on the given physical parameters.

The Maxwell equation (20.3) for the y-component of the current density J gives to $O(\varepsilon^5)$:

$$(1+m)[1-\frac{a^2}{U_0^2})\, B'^{(2)} \mp B'^{(1)}] = -J^{(2)} \tag{29.1}$$

where from (4) the (-) sign refers to supercritical wave speed and the (+) sign refers to subcritical wave speed; and

$$J^{(2)} = \iiint (V \sin\theta)(f_5^+ - f_5^-) V \, dV \, d\theta \, dw \tag{29.2}$$

$$= \iiint (V \cos\theta)\left(\frac{\partial f_5^+}{\partial\theta} - \frac{\partial f_5^-}{\partial\theta}\right) V \, dV \, d\theta \, dw$$

$$= \frac{1}{2}\frac{\partial}{\partial x} \iiint \Bigg\{ [A_{41}^+ VW + (A_{40}^+ + \frac{A_{42}^+}{2}) V^2 \cos\alpha]$$

$$- [A_{41}^- VW + (A_{40}^- + \frac{A_{42}^-}{2}) V^2 \cos\alpha] \Bigg\} V \, dV \, d\theta \, dw$$

$$- B^{(1)} J^{(1)} \cos\alpha$$

$$= s_1 B'''^{(1)} + s_2 B^{(1)} B'^{(1)} + s_3 E''^{(1)} + s_4 b''^{(1)}$$

$$+ \Bigg\{ B'^{(2)} [-(1+m) + (\frac{\phi_1^+}{\gamma_+} + m \frac{\phi_1^-}{\gamma_-}) \cos^2\alpha]$$

$$- E_{''}^{(2)} (\phi_1^+ + \phi_1^-) \frac{\cos\alpha}{\sin\alpha} \Bigg\} ,$$

where s_1, s_2, s_3, and s_4 depend on the given physical parameters. In the calculation of $J^{(2)}$ the explicit need for f_5 was circumvented by taking the $(V \cos\theta)$-moment of the Vlasov equation (9.1) and using the following moment relation (obtained by an integration by parts):

$$\iiint (V \cos\theta) \frac{\partial f}{\partial\theta} V \, dV \, d\theta \, dw$$

$$= \iiint (V \sin\theta) f V \, dV \, d\theta \, dw \, . \tag{29.3}$$

As shown by (29.2), $(\partial f_5/\partial\theta)$ can be expressed in terms of f_k with $k \leq 4$.

The elimination of second order quantities ($E^{(2)}$, $b^{(2)}$,

$B^{(2)}$) from (27), (28), and (29) is easily described, recalling the definitions of $E_{\shortparallel}$ and $\Phi_{\shortparallel}$ from (9.2) and (15.2): 1) Clearly, the second order terms can be eliminated between (27) and (28) since the expressions in curly brackets are identical; 2) taking the derivative of (27) and eliminating $E_{\shortparallel}^{(2)}$ using (29), the coefficient of the remaining second order term in $B'^{(2)}$ becomes zero by the consistency condition (25); and 3) the third independent equation for ($E^{(1)}$, $b^{(1)}$, $B^{(1)}$) is (22.3). From these three differential equations we can obtain a single third-order nonlinear differential equation satisfied by $B^{(1)}$ of the form

$$RB'''^{(1)}(x) + (\mp 1 + SB^{(1)}(x))B'^{(1)}(x) = 0 \tag{30.1}$$

where the coefficients R and S are given by

$$(1+m)R = \tag{30.2}$$

$$= (1+m^3) + \frac{1}{8}\left(\frac{1}{\gamma_+} + \frac{m^3}{\gamma_-}\right)(-11 + 23\sin^2\alpha) + \frac{3}{4}\left(\frac{\phi_1^+}{\gamma_+^2} + m^3\frac{\phi_1^-}{\gamma_-^2}\right)\cos^4\alpha]$$

$$+ \Gamma\cos^2\alpha[-(1-m^2) + \frac{3}{4}\left(\frac{\phi_1^+}{\gamma_+} - \frac{m^2\phi_1^-}{\gamma_-}\right)\cos^2\alpha + \frac{\Gamma}{4}(\phi_1^+ + m\phi_1^-)\cos^2\alpha]$$

$$+ \frac{-[(1 - m^2 + \frac{1}{4}(\frac{1}{\gamma_+} - \frac{m^2}{\gamma_-})(-3 + 5\sin^2\alpha) - \frac{1}{2}(1+m)\Gamma\cos^2\alpha]^2}{\cos^2\alpha\left\{(1+m) + [\left(\frac{\phi_1^+}{\gamma_+} + \frac{m\phi_1^-}{\gamma_-}\right) + \frac{\Gamma}{2}(\phi_1^+ - \phi_1^-)]\sin^2\alpha\right\}}$$

$$(1+m)S = \tag{30.3}$$

$$\cos\alpha\Big\{3(1+m)-[(\frac{\phi_1^+}{\gamma_+}+\frac{m\phi_1^-}{\gamma_-})+\frac{\Gamma}{2}(\phi_1^+-\phi_1^-)]\cos^2\alpha$$

$$-(\frac{\phi_1^+}{\gamma_+}+\frac{m\phi_1^-}{\gamma_-})(2-5\sin^2\alpha)+3(\frac{\phi_2^+}{\gamma_+}+m\frac{\phi_2^-}{\gamma_-})\cos^2\alpha$$

$$-\frac{3}{2}(\frac{\phi_3^+}{\gamma_+^2}+\frac{m\phi_3^-}{\gamma_-^2})\sin^2\alpha\cos^2\alpha-\Gamma(\phi_1^+-\phi_1^-)(1-\frac{5}{2}\sin^2\alpha)$$

$$+\Gamma(\phi_2^+-\phi_2^-)\cos^2\alpha+\frac{\Gamma}{2}(\frac{\phi_3^+}{\gamma_+}-\frac{\phi_3^-}{\gamma_-})\sin^2\alpha\cos^2\alpha$$

$$-\frac{\Gamma^2}{4}[(\phi_2^++m^{-1}\phi_2^-)-3(\phi_3^++m^{-1}\phi_3^-)]\sin^2\alpha\cos^2\alpha$$

$$-\frac{\Gamma^3}{4}[(\gamma_+\phi_2^+-m^{-2}\gamma_-\phi_2^-)-(\gamma_+\phi_3^+-m^{-2}\gamma_-\phi_3^-)\sin^2\alpha\cos^2\alpha\Big\}$$

where ϕ_p is defined by (24.2) and specifically* $\phi_2 = -2\kappa^{-2}(1-\phi_1)$ and $\phi_3 = -\kappa^{-2}(\phi_1-\phi_2)$ and

$$\Gamma = \frac{(\phi_1^+ - \phi_1^-)}{[\gamma_+(1-\phi_1^+)+m^{-1}\gamma_-(1-\phi_1^-)]} \tag{30.4}$$

---

*A preliminary study based on suggestions by C. S. Gardner (private communication) indicate that using these relations for ϕ_2 and ϕ_3 together with the principal value integral ϕ_1 in (24.1) is equivalent to taking "cut-off" distribution functions (at $x = \pm\infty$) which eliminate particles which turn around or are trapped by the steady wave.

The differential equation (30) is of the same form as that derived in [1] and [2] for 1) the steady wave moving in a transverse magnetic field and 2) the cold-plasma oblique case [3]. The ($\mp$) signs allow for both super- and subcritical wave speeds. Although the coefficients R and S are lengthy, the transverse limit $\alpha = 0$ and the cold-plasma limit $T_- = T_+ = 0$ are easily checked by inspection.

We have obtained the parametric dependence of R, S, and the wave speed (25) on α, T_-, and T_+ by electronic computation for three interesting cases: $T_- = mT_+$ (cool electrons), $T_- = T_+$ (equal temperature), and $T_+ = 0$ (cold ions) (see Figs. 3, 4, 5, and 6). For $T_- = mT_+$, we have $\gamma_- = \gamma_+ \equiv \gamma$, $\phi_1^- = \phi_1^+ \equiv \phi_1$ and $\Gamma = 0$; for $T_- = T_+$, we have $\gamma_- = m\gamma_+$. In both of these cases the critical transition angle $\alpha_{cr.}$ where the solution changes from a positive pulse moving with supercritical velocity to a negative pulse traveling at subcritical speed increases rapidly as a function of temperature. However, the cold ion case ($T_+ = 0$, $T_- > 0$) yields a singularly different result: $\alpha_{cr.}$ is practically independent of T_-, the transition angle being approximately the cold plasma value found by Morton [3]. The strong dependence of the coefficient R on the ion temperature T_+ is emphasized in Fig. 4; the width of the pulse is proportional to $R^{\frac{1}{2}}$. The coefficient S, which is proportional to the amplitude of the pulse, exhibits some nonuniform behavior for both small and large obliqueness angle α,

particularly with appreciable electron temperature T_- (Fig. 5). This non-uniform behavior of S is also reflected in the wave speed (U_o/a) (Fig. 6). However, for the cool electron case ($T_- = mT_+$) the wave speed transverse to the undisturbed magnetic field ($U_o \cos \alpha/a$) is practically independent of the obliqueness angle α over the calculated range of ion temperatures.

7. Pressure Tensor, P_{ij}.

The pressure tensor for ions (or P^-_{ij} for electrons) is

$$P_{ij} = \iiint_{-\infty}^{\infty} q_{ij} \; f \; du \; dv \; dw \tag{31.1}$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_{-\infty}^{\infty} q_{ij} \; f \; V \; dv \; d\theta \; dw$$

where

$$q_{ij} = \begin{pmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{pmatrix} \tag{31.2}$$

and in cylindrical polar coordinates,

$$u = V \cos \theta \; , \quad v = V \sin \theta \tag{31.3}$$

$$u^2 = \frac{V^2}{2}(1+\cos 2\theta) \;, \quad v^2 = \frac{V^2}{2}(1-\cos 2\theta) \tag{31.4}$$

$$uv = \frac{V^2}{2}\sin 2\theta, \quad uw = wV\cos\theta, \quad vw = wV\sin\theta \tag{31.5}$$

The power series expansion in ε for the pressure tensor is

$$P_{ij} = P\cdot\delta_{ij} + \varepsilon^2 P_{ij}^{(2)} + \varepsilon^3 P_{ij}^{(3)} + \varepsilon^4 P_{ij}^{(4)} + O(\varepsilon^5) \tag{32.1}$$

where P is the dimensionless scalar pressure

$$P = \frac{1}{3}\iiint (V^2 + w^2)\Psi\, V\, dV\, d\theta\, dw \tag{32.2}$$

$$= kT/m_+U_0^2 = 1/2\gamma_+$$

(cf. eq. (13.2)) and the perturbed pressure tensor is calculated using the series solution for f given by f_k in (13) to (18) and calculating the moments (31). We summarize the results for $P_{ij}^{(k)}$: $P_{ij}^{(2)}$ is given by $P_{ij}^{(2)} = 0$ for $i \neq j$ and the diagonal elements are

$$P_{11}^{(2)} = P_{22}^{(2)} = \frac{1}{2}\iiint A_{20}V^3 dV\, d\theta\, dw \tag{33.1}$$

$$= \frac{B^{(1)}\phi_1 \cos\alpha}{\gamma} + \frac{\Phi_{"}^{(1)}(1-\phi_1)}{\sin\alpha}$$

$$P_{33}^{(2)} = \iiint A_{20} w^2 V \, dV \, d\theta \, dw \tag{33.2}$$

$$= \frac{1}{\sin^3\alpha} [B^{(1)}(\phi_1 - 1)\sin\alpha\cos\alpha + \Phi_{\parallel}^{(1)}\sin^2\alpha$$

$$+ 2\gamma\Phi_{\parallel}^{(1)}(1 - \phi_1)]$$

$P_{ij}^{(2)}$ is directly proportional to the field $B^{(1)}$ (recalling (21.3)) and comparable to the usual guiding center theory. $P_{ij}^{(3)}$ has both right and left diagonal elements equal to zero, i.e., $P_{ij}^{(3)} = 0$ for $i = j$ and/or $(i+j) = 4$ and the non-diagonal elements are

$$P_{12}^{(3)} = P_{21}^{(3)} = \frac{1}{4} \iiint B_{32} V^3 dV \, d\theta \, dw \tag{34.1}$$

$$= -\frac{1}{4} \frac{B'^{(1)}\cos\alpha}{\gamma}$$

$$P_{23}^{(3)} = P_{32}^{(3)} = \frac{1}{2} \iiint B_{31} w V^2 dV \, d\theta \, dw \tag{34.2}$$

$$= -\frac{1}{2} \frac{B'^{(1)}}{\gamma\sin\alpha} [\sin^2\alpha + (\gamma\Gamma - 2)\cos^2\alpha]$$

$P_{ij}^{(3)}$ is directly proportional to the field gradient $dB^{(1)}/dx$. $P_{ij}^{(4)}$ has non-zero right and left diagonal elements and $P_{ij}^{(4)} = 0$ for $i \neq j$ and $(i+j) \neq 4$; $P_{ij}^{(4)}$ contains second order field quantities ($E^{(2)}$, $b^{(2)}$, $B^{(2)}$) and is rather lengthy so only the integral form of the components are given

$$P_{11}^{(4)} = P_{22}^{(4)} = \frac{1}{4} \iiint (2A_{40}V^3 + A_{42}V^3) dV\, d\theta\, dw \qquad (35.1)$$

$$P_{33}^{(4)} = \iiint A_{40} w^2 V\, dV\, d\theta\, dw \qquad (35.2)$$

$$P_{13}^{(4)} = P_{31}^{(4)} = \frac{1}{2} \iiint A_{41} w V^2 dV d\theta dw \qquad (35.3)$$

Thus to $O(\varepsilon^4)$ the pressure tensor P_{ij} is filled with <u>symmetrical non-zero</u> elements.

8. Comments on Singularities in Velocity Space

We can discuss the singularities in velocity space only briefly here. We begin the discussion by assuming that the equilibrium distribution function f_0 <u>and</u> the first-order distribution function f_1 are spatially uniform so that there are no charge <u>nor</u> current densities in the equilibrium plasma. Then,

$$f_0 = A_{00} = \Psi(V^2 + w^2) \qquad (36)$$

$$f_1 = A_{10} = 0 \qquad (37)$$

$$\begin{aligned} f_2 &= A_{21} \cos\theta + A_{20}^{*} \qquad (38) \\ &= 2B^{(1)}\, \Psi'\, V \cos\theta + \chi_2(x,V,w) \end{aligned}$$

where $A^*_{20} \equiv \chi_2$ is the θ-integration variable. In order to determine χ_2 we impose the condition that to next order f_3 is continuous in θ, i.e., using the Vlasov equation (9) and

$$\int_0^{2\pi} \frac{\partial f_3}{\partial \theta} \, d\theta = 0 \tag{39}$$

and we get

$$(1 + w \sin \alpha) \frac{\partial \chi_2}{\partial x} = -B'^{(1)} \Psi' V^2 \cos \alpha - 2E_{\parallel}^{(1)} \Psi' w \; . \tag{40.1}$$

Integrating χ_2 using the boundary condition $f_2 = 0$ at $x = -\infty$ we get

$$(1 + w \sin \alpha) \chi_2 = -B^{(1)} \Psi' V^2 \cos \alpha - 2\Phi_{\parallel}^{(1)} \Psi' w \; . \tag{40.2}$$

Since the range of integration for w, when taking moments, is $-\infty \leq w \leq \infty$, the quantity $W \equiv (1 + w \sin \alpha)$ can be zero. Hence we introduce a δ-function in velocity space and write χ_2 as

$$A^*_{20} = \chi_2 = - \frac{\Psi'}{W} \left(B^{(1)} V^2 \cos \alpha + 2\Phi_{\parallel}^{(1)} w\right)$$
$$+ B^{(1)}(x) \cdot C(V,w) \cdot \delta(W) \tag{41}$$

for $\alpha > 0$ and $C(V,w)$ is an undetermined regular function. The difference between A^*_{20} in (41) and A_{20} in (15.2) is the

δ-function term. We take the δ-function term to be proportional to the perturbed field quantity $B^{(1)}$; otherwise the speed of the wave $(U_0\ a)$, cf. (25), will be x-dependent. But now the algorithm for the asymptotic solution given in the previous sections does not determine $C(V,w)$ nor two additional regular functions of (V,w) arising from the higher order singularities when $W^2 = 0$ and $W^3 = 0$ in f_4, cf. (17). Thus there is a <u>parametric</u> non-uniqueness in the determination of both the speed and structure of the steady wave in terms of the given physical parameters (α, m, T_-, T_+), i.e., in the determination of (U_0/a) and the coefficients R and S of the third order differential equation (30). Although the present calculations apparently can be interpreted in terms of cut-off (or cut-out) distribution functions, a more precise analysis seems desirable in the future--and possible (see footnote, page 21).

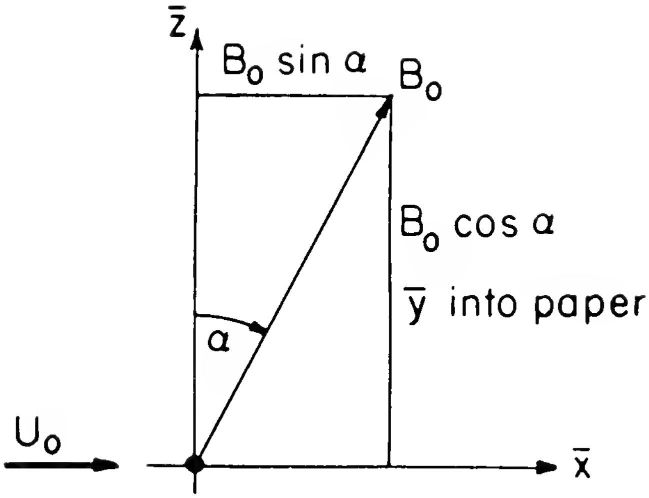

Figure 1. Configuration Space at $x = -\infty$

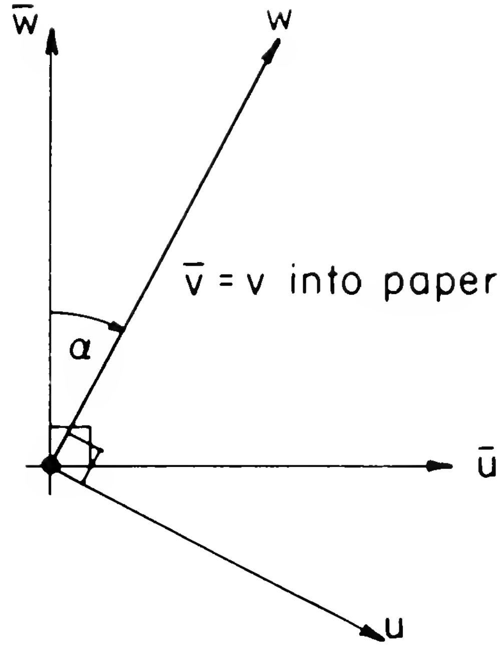

Figure 2a. Velocity Space: Rotation by α

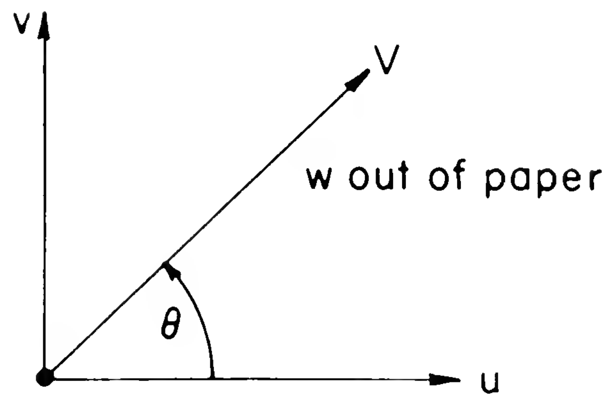

Figure 2b. Velocity Space: Transverse Plane to w

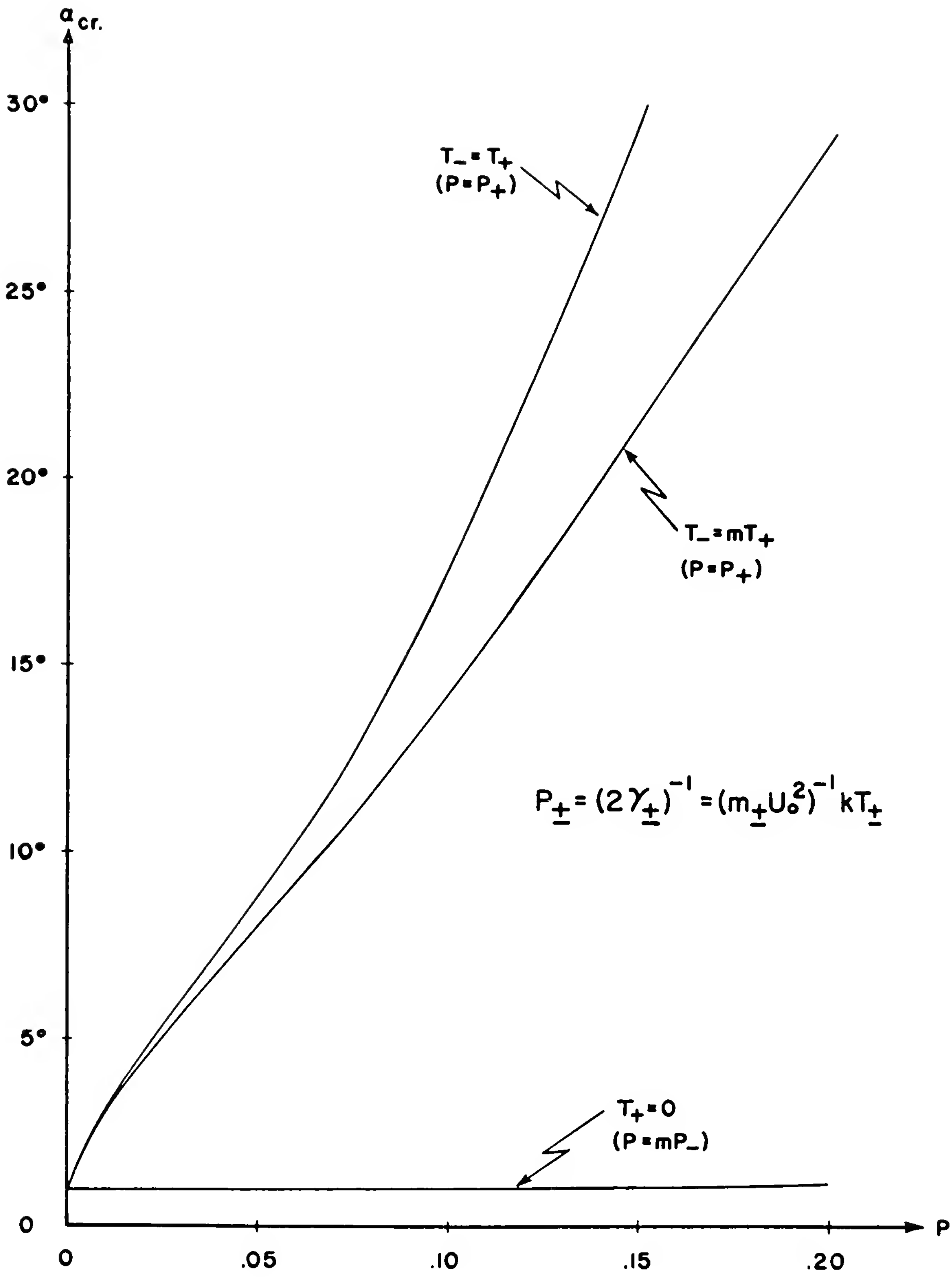

FIGURE 3. CRITICAL OBLIQUENESS ANGLE vs. PRESSURE FOR COEFFICIENT R=0.

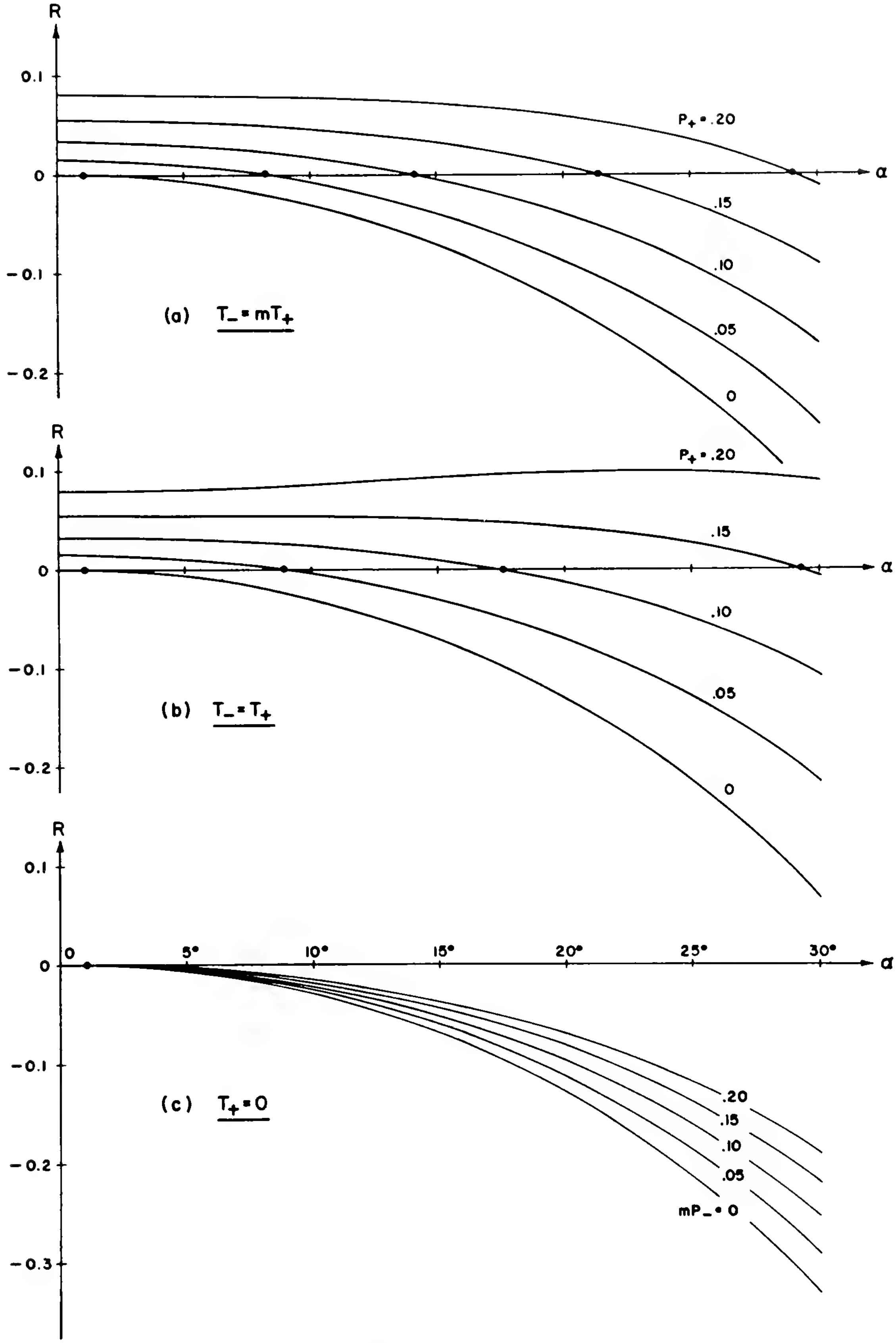

FIGURE 4. COEFFICIENT R vs. OBLIQUENESS ANGLE α

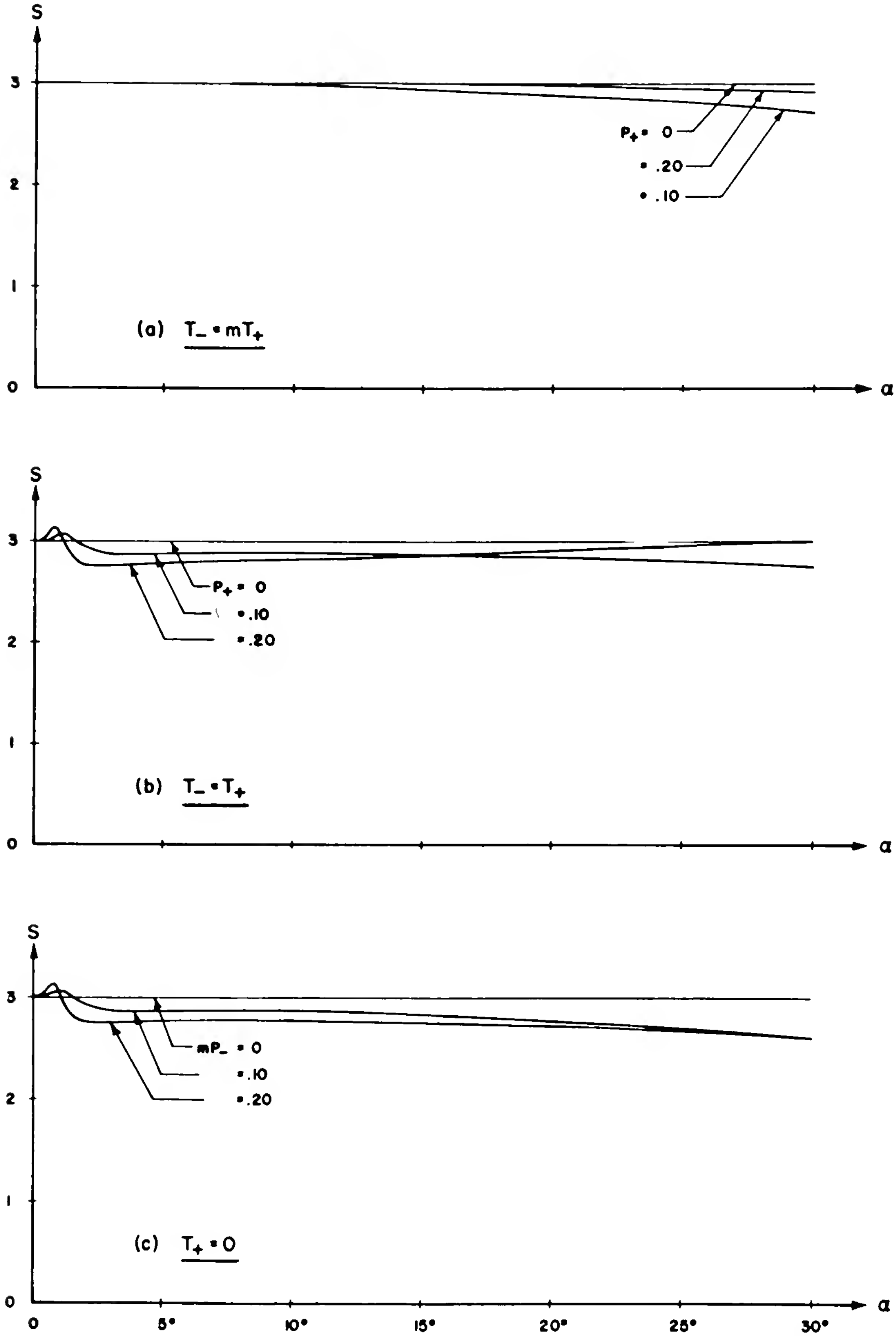

FIGURE 5. COEFFICIENT S vs. OBLIQUENESS ANGLE α

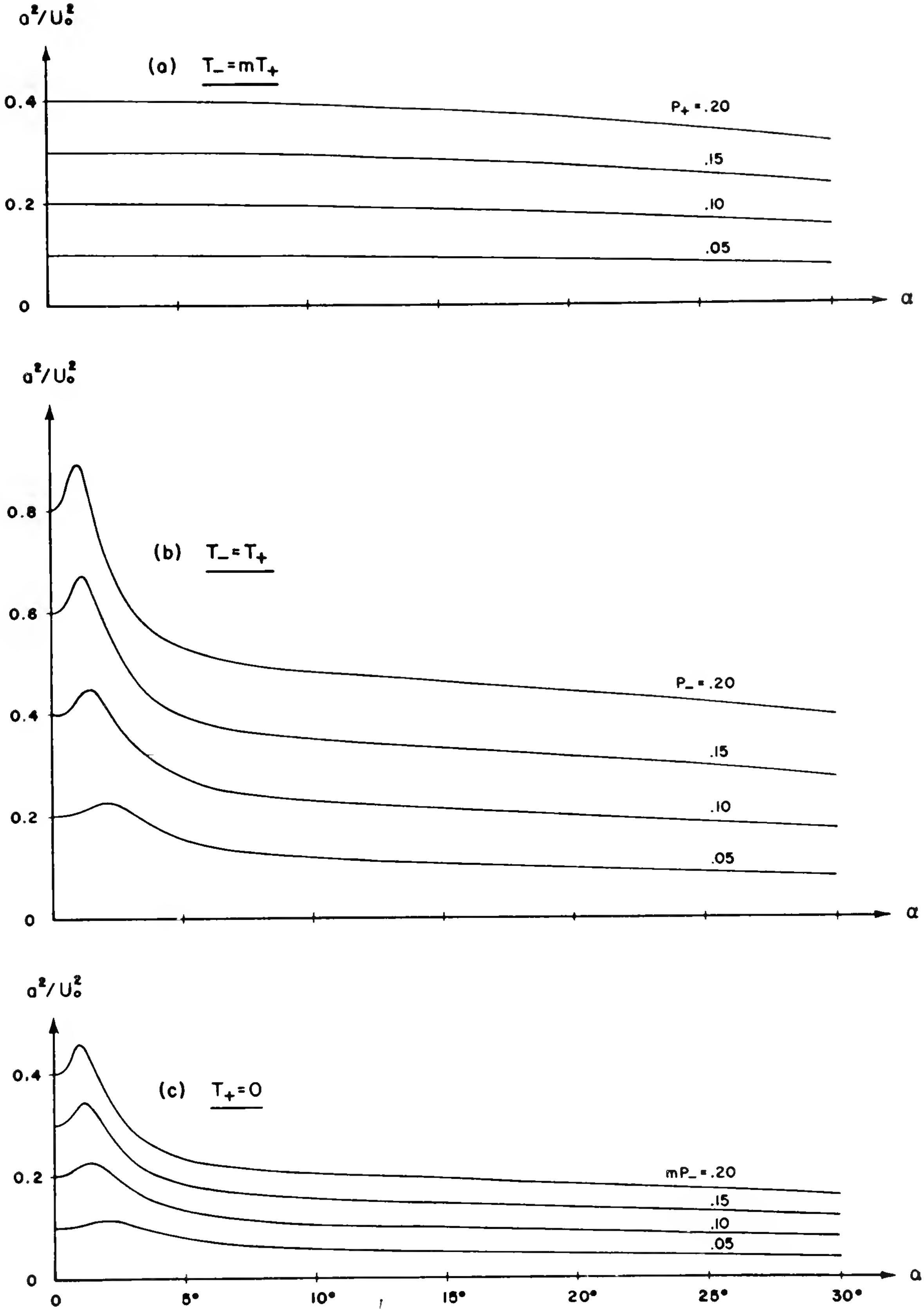

FIGURE 6. (WAVE SPEED)$^{-2}$ vs. OBLIQUENESS ANGLE α

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